# Nonexistence of Finite-dimensional Quantizations of a Noncompact Symplectic Manifold

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**Abstract** *We prove that there is no faithful finite-dimensional representation by skew-hermitian matrices of a "basic algebra of observables" $\mathcal{B}$ on a noncompact symplectic manifold $M$. Consequently there exists no finite-dimensional quantization of* any *Lie subalgebra of the Poisson algebra $C^\infty(M)$ containing $\mathcal{B}$.*

## 1. Introduction

Let $M$ be a connected noncompact symplectic manifold. On physical grounds one expects a quantization of $M$, if it exists, to be infinite-dimensional. This is what we rigorously prove here, in the framework of the paper [GGT]. Our precise hypotheses are spelled out below.

A key ingredient in the quantization process is the choice of a *basic set of observables* in the Poisson algebra $C^\infty(M)$. This is a finite-dimensional linear subspace $\mathcal{B}$ of $C^\infty(M)$ such that

(B1) (*Completeness*) the Hamiltonian vector fields $X_f$ , $f \in \mathcal{B}$, are complete,

(B2) (*Transitivity*) $\{X_f \mid f \in \mathcal{B}\}$ spans $TM$, and

(B3) (*Minimality*) $\mathcal{B}$ is minimal with respect to these conditions.

In addition to these conditions we assume in this paper that $\mathcal{B}$ forms a Lie algebra under the Poisson bracket. We then refer to $\mathcal{B}$ as a *basic algebra*. (Note also that unlike in [GGT], we do not require here that $1 \in \mathcal{B}$. )

Now fix a basic algebra $\mathcal{B}$, and let $\mathcal{O}$ be any Lie subalgebra of $C^\infty(M)$ containing $1$ and $\mathcal{B}$. Then by a *finite-dimensional quantization* of the pair $(\mathcal{O}, \mathcal{B})$ we mean a Lie representation $\mathcal{Q}$ of $\mathcal{O}$ by skew-hermitian matrices on $\mathbb{C}^n$ such that

(Q1) $\mathcal{Q}(1) = I$ ,

(Q2) $\mathcal{Q}(\mathcal{B})$ is irreducible, and

(Q3) $\mathcal{Q}$ is faithful on $\mathcal{B}$.

We refer the reader to [GGT] for a detailed discussion of these matters. We remark that in the infinite-dimensional case there are additional conditions which must be imposed upon a quantization. We also elaborate briefly on (Q3). Although faithfulness is not usually assumed in the definition of a quantization, it seems to us a reasonable requirement in that a classical observable can hardly be regarded as "basic" in a physical sense if it is in the kernel of a quantization map. In this case, it cannot be obtained in any classical limit from the quantum theory.

## 2. The Obstruction

Given the definitions above, we state our result:

**Theorem.** *Let* $M$ *be a noncompact symplectic manifold,* $\mathcal{B}$ *a basic algebra on* $M$ *, and* $\mathcal{O}$ *any Lie subalgebra of* $C^\infty(M)$ *containing* $\mathcal{B}$ *. Then there is no finite-dimensional quantization of* $(\mathcal{O}, \mathcal{B})$ *.*

As the proof will show, we do not need conditions (Q1) or (Q2) to obtain the theorem. Moreover, the subalgebra $\mathcal{O}$ is irrelevant since the proof depends only on the Lie theoretical properties of the basic algebra $\mathcal{B}$ and its action on $M$ .

**Proof:** We argue by contradiction. Suppose there exists a finite-dimensional quantization $\mathcal{Q}$ of the basic algebra $\mathcal{B}$ . Since $\mathcal{Q}(\mathcal{B})$ consists of skew-hermitian matrices, it is completely reducible. Since $\mathcal{Q}$ is faithful, one deduces from [V, Thm 3.16.3] that $\mathcal{B}$ is reductive, i.e. $\mathcal{B} = \mathfrak{s} \oplus \mathfrak{z}$ where $\mathfrak{s}$ is semisimple and $\mathfrak{z}$ is the center of $\mathcal{B}$ . We show that $\mathfrak{z} = \{0\}$ . Indeed, by the transitivity condition (B2), the elements of $\mathfrak{z}$ must be constant but, if these are nonzero, then $\mathfrak{s}$ alone would serve as a basic algebra, contradicting the minimality condition (B3). Thus $\mathfrak{z} = \{0\}$ and $\mathcal{B} = \mathfrak{s}$ is semisimple.

Let $B$ be the connected, simply connected Lie group with Lie algebra $\mathcal{B}$ . We show that $B$ is noncompact. Let $\mathfrak{g}$ be the Lie algebra $\{X_f \mid f \in \mathcal{B}\}$ . By (B1) the vector fields in $\mathfrak{g}$ are complete and so by [V, Thm. 2.16.13] this infinitesimal action of $\mathfrak{g}$ can be integrated to an action of the connected, simply connected Lie group $G$ with Lie algebra $\mathfrak{g}$ . Condition (B2) implies that this action is locally transitive and thus globally transitive as $M$ is connected. Thus the noncompact manifold $M$ is a homogeneous space for $G$ , and so $G$ must be noncompact as well. Now $\mathcal{B}$ is isomorphic either to $\mathfrak{g}$ or to a central extension of $\mathfrak{g}$ by constants. Since $\mathcal{B}$ is semisimple, the latter alternative is impossible. Hence $B$ is isomorphic to $G$ and so is noncompact.

Now consider a unitary representation $\pi$ of $B$ on $\mathbb{C}^n$ . Decompose $B$ into a product $B_1 \times \cdots \times B_K$ of simple groups. Then (at least) one of these, say $B_1$ , must be

noncompact. But it is well-known that a connected, simple, noncompact Lie group has no nontrivial unitary representations [BR, Thm. 8.1.2]. Thus $\pi|B_1$ is trivial, i.e. $\pi(b) = I$ for all $b \in B$. Since every finite-dimensional quantization $\mathcal{Q}$ of $\mathcal{B}$ is a derived representation of some unitary representation $\pi$ of $B$, it follows that $\mathcal{Q}|\mathcal{B}_1 = 0$, and so $\mathcal{Q}$ cannot be faithful. ■

## 3. Discussion

This theorem is complementary to a recent result of [GGG] which states that there are no nontrivial quantizations (finite-dimensional or otherwise) of $(P(\mathcal{B}), \mathcal{B})$ on a *compact* symplectic manifold $M$, where $P(\mathcal{B})$ is the Poisson algebra of polynomials generated by the basic algebra $\mathcal{B}$. The proof of that result leaned heavily on the algebraic structure of $P(\mathcal{B})$; indeed, when $M$ is compact, it turns out that $\mathcal{B}$ must be compact semisimple, and such algebras *do* have faithful finite-dimensional representations by skew-hermitian matrices. Thus in the compact case, the obstruction to the existence of a quantization is Poisson, rather than Lie theoretical. Combining [GGG] with the present theorem, we can now assert, roughly speaking, that no symplectic manifold with a basic algebra has a finite-dimensional quantization.

We hope to address the issue of whether there are obstructions in general to infinite-dimensional quantizations of noncompact symplectic manifolds in future work. Certainly such obstructions exist in specific examples, such as $\mathbb{R}^{2n}$ [GGT] and $T^*S^1$ [GG]. This appears to be a difficult problem, however.


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